Mathematical Finance
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Assignment 12

Long-term forward rates never fall (see Dybvig, Ingersoll Jr., and Ross [1])

We place ourselves in the context of Chapter 9 in the notes, and we assume that the day-count convention has the property that for any $0 \leq t \leq T$

$$\tau(t,T) = T - t.$$

1) Prove that the instantaneous forward rate can be used to express zero-coupon bond prices as

$$B(t,T) = \exp\bigg(- \int_t^T f(t,s)\mathrm{d}s \bigg),\ 0 \leq t \leq T.$$

The goal of this exercise to study the so-called long-term forward rate defined, when it exists, by

$$f_L(t) := \lim_{T\to+\infty} f(t,T),\ t \geq 0,$$

and prove that it is always a non-decreasing function of time.

2) Prove that for any $0 \leq t \leq T$, we have

$$\inf_{t\leq s\leq T} f(t,s) \leq r(t,T) \leq \sup_{t\leq s\leq T} f(t,s).$$

Prove that if, for some $t \geq 0$, $f_L(t)$ exists, then the continuously-compounded long-term rate $r_L(t) := \lim_{T\to+\infty} r(t,T)$ also exists, and we have $r_L(t) = f_L(t)$.

3) Define for any $t \geq 0$, when the limit exists, the annually-compounded long-term rate

$$y_L(t) := \lim_{T\to+\infty} B(t,T)^{-\frac{1}{T}} - 1.$$

Prove that $r_L(t)$ exists if and only if $y_L(t)$ exists, and give a relationship between these two quantities.

4) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of non-negative random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G}$ be a sub–σ-algebra of $\mathcal{F}$. Assume that X_n converges with $\mathbb{P}$-probability 1 to some random variable X, as n goes to $+\infty$, and that

$$Y := \liminf_{n\to+\infty} \mathbb{E}^{\mathbb{P}}\big[(X_n)^n \big| \mathcal{G}\big]^{\frac{1}{n}} < +\infty,\ \mathbb{P}\text{–a.s.}$$

Prove that for any bounded, non-negative random variable Z, we have

$$\mathbb{E}^{\mathbb{P}}[XZ] \leq \mathbb{E}^{\mathbb{P}}[YZ],$$

and deduce that, with $\mathbb{P}$-probability 1, $X \leq Y$.

Hint: If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two (possibly random) sequences of non-negative numbers such that $\lim_{n\to+\infty} y_n$ exists, then prove first that[1]

$$\liminf_{n\to+\infty} (x_n y_n) \leq \liminf_{n\to+\infty} x_n \lim_{n\to+\infty} y_n.$$

Hölder's inequality and Fatou's lemma should then help.

5) Prove that if $0 \leq s \leq t$ are such that both $y_L(t)$ and $y_L(s)$ exist, then $y_L(s) \leq y_L(t)$.

[1] We recall that for a sequence $(u_n)_{n\in\mathbb{N}}$, we have

$$\liminf_{n\to+\infty} u_n := \sup_{n\in\mathbb{N}} \inf_{m\geq n} u_m.$$

It could be useful to prove first that for any $T \geq t$

$$B(s,T) = B(s,t)\mathbb{E}^{\mathbb{P}_t}[B(t,T)|\mathcal{F}_s].$$

6) Deduce that if $0 \leq s \leq t$ are such that both $f_L(t)$ and $f_L(s)$ exist, then $f_L(s) \leq f_L(t)$.

Options on forward contracts in Vašíček's model

We consider Vašíček's model for the short-rate r

$$r_t = r_0 + \int_0^t k(\theta - r_s)\mathrm{d}s + \sigma W_t^\lambda,\ t \geq 0,\ \mathbb{P}\text{–a.s.} \tag{0.1}$$

And we denote as usual by

$$b(t,T) := -\sigma \frac{1-\mathrm{e}^{-k(T-t)}}{k},\ 0 \leq t \leq T,$$

the volatility of a zero-coupon bond with maturity T in this model.

1) For any $0 \leq T \leq s$, we define

$$g(T,s) := B(T,s)^{-1}.$$

Prove that for any $0 \leq t \leq T \leq s$

$$\mathrm{d}g(t,s) = -g(t,s)\big(\big(r_t + b(t,s)(b(t,T) - b(t,s))\big)\mathrm{d}t + b(t,s)\mathrm{d}W_t^T\big),$$

and deduce that

$$g(T,s) = g(t,s)\exp\bigg(-\int_t^T \bigg(r_u + b(u,s)(b(u,T) - b(u,s)) + \frac{1}{2}b^2(u,s)\bigg)\mathrm{d}u - \int_t^T b(u,s)\mathrm{d}W_u^T\bigg).$$

Using the explicit formula for $\int_t^T r_u \mathrm{d}u$ in Vašíček's model, deduce finally that

$$g(T,s) = g(t,s)\exp\bigg(\frac{b(t,T)}{\sigma}(r_t - \theta) - \theta(T-t) - \int_t^T \bigg(b(u,s)(b(u,T) - b(u,s)) + \frac{1}{2}b^2(u,s) - b^2(u,T)\bigg)\mathrm{d}u\bigg)$$
$$\times \exp\bigg(-\int_t^T \big(b(u,s) - b(u,T)\big)\mathrm{d}W_u^T\bigg).$$

2) For any $0 \leq t \leq T$, compute explicitly the forward price at time t, for the maturity T, of the payoff $g(T,s)$, that is to say $F_t\big(T; g(T,s)\big)$. You should express your answer as

$$F_t\big(T; g(T,s)\big) = B(t,s)^{-1}\exp\big(\alpha(t,T)r_t + \beta(t,T,s)\big),$$

where you will give explicitly the deterministic maps α and β.

Hint: Recall that for any contingent claim ξ with maturity T (that is to say which is $\mathcal{F}_T$-measurable), we have

$$F_t(T;\xi) = \mathbb{E}^{\mathbb{P}_T}[\xi|\mathcal{F}_t].$$

3) What is the dynamics under $\mathbb{P}_T$ of $\big(F_t\big(T; g(T,s)\big)\big)_{t\in[0,T]}$? Recall that you already know that $\big(F_t\big(T; g(T,s)\big)\big)_{t\in[0,T]}$ is an $(\mathbb{F}, \mathbb{P}_T)$-martingale.

4) Deduce the price at any time $t \in [0,T]$ of a call option with maturity T, strike K, and underlying $F_\cdot(T, g(T,s))$, that is to say the option with payoff at time T equal to $\big(F_T\big(T; g(T,s)\big) - K\big)^+$. You will express the result as a Black–Scholes like formula.

References

[1] P.H. Dybvig, J.E. Ingersoll Jr., and S.A. Ross. Long forward and zero–coupon rates can never fall. *Journal of Business*, 69(1):1–25, 1996.